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An Odd Solution to the Functional Equation $P\left(\frac{x+1}{2}\right)=\exp(P(x))$

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I. Introduction

The purpose of this paper is to study the system of functional equations,

$$P\left(\frac{x+1}{2}\right)=\exp(P(x)) \tag{1}$$

$$P(x)+P(-x)=0, \tag{2}$$

where P is a real function defined on the open interval $(-1, 1)$. The following theorem will be proved:

THEOREM. *The system of functional equations* (1) *and* (2) *has a unique continuous solution. This solution is strictly increasing and has a zero derivative at every dyadic rational in* $(-1, 1)$.

The system of functional equations (1) and (2) arises naturally in the following manner: Consider a homeomorphism Φ from the square $(-1,1)\times(-1,1)$ to the plane $X\times Y$ which treats each co-ordinate in the same fashion and preserves symmetry about the Y-axis. Such a Φ must have the form

$$\Phi((x, y))=(P(x), P(y)),$$

where P is a homeomorphism from the interval $(-1, 1)$ to the real line; the symmetry condition implies that

$$\Phi((-x, y))=(P(-x), P(y))=(-P(x), P(y)),$$

whence P must be an odd function, i.e., satisfy (2). If one now asks for the function Φ which maps the line segment

$$y=\left(\frac{x+1}{2}\right), \ -1<x<1,$$

onto the exponential curve $y=e^x$, $-\infty<x<\infty$, then Φ must take the point $(x, ((x+1)/2))$ on the line segment to the point $(P(x), P((x+1)/2))$ on the exponential curve, whence P must satisfy (1).

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II. Proof of Theorem

The proof of the theorem stated above will be divided into a sequence of lemmas.

LEMMA 1. *If P is a function satisfying* (1) *and* (2) *then P also satisfies*

$$P(t)\,P(1-t)=1, \tag{3}$$

$$P\left(\frac{t}{2}\right)=\exp(-1/P(t)), \quad \text{for} \quad 0<t<1. \tag{4}$$

Proof. Let $0<t<1$. Then, using (1) and (2),

$$\begin{aligned} 1=\exp(0) &= \exp(P(2t-1))\cdot\exp(-P(2t-1)) \\ &= \exp(P(2t-1))\cdot\exp(P(1-2t)) \\ &= P\left(\frac{(2t-1)+1}{2}\right)\cdot P\left(\frac{(1-2t)+1}{2}\right) \\ &= P(t)\cdot P(1-t). \end{aligned}$$

This proves (3). Now, using (1), (2), and (3),

$$P\left(\frac{t}{2}\right)=\frac{1}{P\left(1-\frac{t}{2}\right)}=\frac{1}{P\left(\frac{(1-t)+1}{2}\right)}=\frac{1}{\exp(P(1-t))}=\frac{1}{\exp(1/P(t))}=\exp\left(\frac{-1}{P(t)}\right).$$

The identities (1)–(4) determine P recursively on the dyadic rationals in $(-1, 1)$ as follows:

(i) From (2) we have that $P(0)=0$ and from (1) that $P(1/2)=\exp(0)=1$. Thus P is determined on $D_1=\{0, 1/2\}$.

(ii) Now suppose that the values of P have been determined on the set

$$D_n=\left\{\frac{m}{2^n} \mid m=0,1,2,\ldots,2^n-1\right\}.$$

Then (4) determines the values of P on

$$A_{n+1}=\left\{\frac{m}{2^{n+1}} \mid m=0,1,2,\ldots,2^n-1\right\}$$

and (1) determines P on

$$B_{n+1}=\left\{\frac{m+2^n}{2^{n+1}} \mid m=0,1,2,\ldots,2^n-1\right\}.$$

Thus P is determined on $D_{n+1}=A_{n+1}\cup B_{n+1}$. But $\bigcup_{n=1}^{\infty} D_n$ is just the set of dyadic rationals in $[0, 1)$. Thus P is defined on the dyadic rationals in $[0, 1)$ and, by (2), on D, the set of dyadic rationals in $(-1, 1)$.

Let the binary expansion of a dyadic rational $d\in(0, 1)$ be given by

$$d=\sum_{k=1}^{n} \frac{\varepsilon_k}{2^k} = .\varepsilon_1\varepsilon_2 \ldots \varepsilon_n \tag{5}$$

where $\varepsilon_k=0$ or 1, $k=1, 2,\ldots, n-1$, and $\varepsilon_n=1$. Then $P(d)$ may be computed by means of the following algorithm:

Let e^+ and e^- denote, respectively, the exponential and negative exponential function, i.e. $e^+(x)=\exp(x)$ and $e^-(x)=\exp(-x)$; and let any finite string of e^+'s and e^-'s denote the value of the corresponding composite function at 0, so that, for exemple

$$e^+e^+e^-e^+e^- = \exp(\exp(\exp(-\exp(\exp(-0))))).$$

Then, since $P(0)=0$, we find by (1) and (4)

$$P(.1)=1=e^+$$
$$P(.01)=e^-e^-, \qquad P(.11)=e^+e^+$$
$$P(.001)=e^-e^+e^-, P(.011)=e^-e^-e^+, P(.101)=e^+e^-e^-, P(.111)=e^+e^+e^+.$$

In each of these evaluations, the first function in a composition is arbitrary since $e^+(0)=e^-(0)$. The choice of the first function in the composition is made here for consistency with the following algorithm. In general for a binary expansion, d, of length n, as given by (5), $P(d)$ is determined as follows: In the binary expansion of d,

(a) Replace ε_1 by e^+ if $\varepsilon_1=1$ and by e^- if $\varepsilon_1=0$

(b) Replace ε_{k+1} by e^+ if $\varepsilon_{k+1}=\varepsilon_k$ and by e^- if $\varepsilon_{k+1}\neq\varepsilon_k$

(i.e. replace each binary digit after the first by e^+ or e^- according as it does or does not match the digit to its left)

To prove that this algorithm does indeed give the correct values for $P(d)$, we first note that it works for all binary expansions of length 1, i.e. $P(.1)=1$. Now suppose it works for all binary expansions of length at most n, i.e. for all elements of D_n in (ii) above. Let $d\in D_{n+1}=A_{n+1}\cup B_{n+1}$. If $d\in A_{n+1}$ then $d=m/2^{n+1}$ for some $(m/2^n)\in D_n$. In this case the binary expansion of d is given by a 0 followed by the digits of the expansion of $m/2^n$. Applying (a) and (b) to the binary expansion of d yields a string of length $n+1$ beginning with e^- followed by an e^- or e^+, according as the first element of the string for $m/2^n$ is e^+ or e^- followed by the remaining $(n-1)$ symbols of the string for $m/2^n$. By (4), this is precisely the string for $\exp(-1/P(m/2^n))$. For example, from

$$P(.1111011) = e^+e^+e^+e^+e^-e^-e^+,$$

we get

$$\begin{aligned} P(.01111011) &= e^-e^-e^+e^+e^+e^-e^-e^+ \\ &= \exp(-1/e^+e^+e^+e^+e^-e^-e^+) = \exp(-1/P(.1111011)). \end{aligned}$$

Thus the algorithm yields the correct value of $P(d)$ for $d \in A_{n+1}$.

Next, if $d \in B_{n+1}$ then $d = m/2^{n+1} + 1/2$ for some $m/2^n \in D_n$ and the binary expansion of d is given by a 1 followed by the digits of the expansion of $m/2^n$. Applying (a) and (b) to the expansion of d yields a string of length $n+1$ beginning with an e^+ followed by the string for $m/2^n$; e.g. from

$$P(.01011) = e^-e^-e^-e^-e^+$$

we get

$$P(.101011) = e^+e^-e^-e^-e^-e^+ = \exp(e^-e^-e^-e^-e^+) = \exp(P(.01011))$$

and by (1), $P(1/2 + m/2^{n+1}) = \exp(P(m/2^n))$. Thus the algorithm also yields the correct value of $P(d)$ for $d \in B_{n+1}$. Thus the algorithm determines the values of P for any dyadic rational in $[0, 1)$. Whence, by (2), it determines the value of P for any dyadic rátional in $(-1, 1)$.

LEMMA 2. *Suppose* $\varepsilon_1\varepsilon_2 \ldots \varepsilon_n$ *gives rise by the algorithm to* $e_1e_2 \ldots e_n$, *where* $e_i = e^+$ or e^-, $i = 1, \ldots, n$. *Then the number of* e^-*'s in the string* $e_1e_2 \ldots e_j$, $j \leqslant n$, *is even or odd according as* $\varepsilon_j = 1$ or 0.

Proof. Use induction on the length of the string.

LEMMA 3. *Let a and b be non-negative and not both zero. Then* $e_1 \circ e_2 \circ \cdots \circ e_j \circ e^+(a)$ *is less than or greater than* $e_1 \circ e_2 \circ \cdots \circ e_j \circ e^-(b)$ *according as the string* $e_1e_2 \ldots e_je^+$ *has an odd or even number of* e^-*'s,* $e_i = e^+$ *or* e^-, $i = 1, 2, \ldots, j$.

Proof. First note that if a and b are non-negative and not both zero then $e^+(a) > e^-(b)$ and that, if $0 \leqslant c < d$, then $e^+(c) < e^+(d)$ and $e^-(c) > e^-(d)$. Thus, if we operate on the inequality $e^+(a) > e^-(b)$ with composite function $e_1 \circ e_2 \circ \cdots \circ e_j$, the sense of the inequality changes whenever an e^- is performed on both sides and remains unchanged otherwise.

LEMMA 4. *The composite function* $e_1 \circ e_2 \circ \cdots \circ e_j \circ e^+$ *is strictly increasing or decreasing on* $[0, 1)$ *according as the string* $e_1e_2 \ldots e_je^+$ *has an even or an odd number of* e^-*'s.*

LEMMA 5. *If P is a function satisfying* (1) *and* (2) *then P is uniquely determined and strictly increasing on the dyadic rationals in* $(-1, 1)$.

Proof. Consider d_1 and d_2, two dyadic rationals in (0, 1) with $d_1 < d_2$. One of two cases is true of their binary expansions:

(i) For some m, the two expansions agree for the first m places to the right of the binary point; the $m+1^{st}$ place of the expansion of d_1 has a 0 and that of d_2 a 1, e.g. $d_1 = .1101101$, $d_2 = .110111$, $m=5$.

(ii) m is the length of d_1 and is less than the length of d_2; and the first m places of d_2 to the right of the binary point agree with d_1, e.g. $d_1 = .1011$, $d_2 = .101101$, $m=4$.

Now let $d_1 = .\varepsilon_1\varepsilon_2 \ldots \varepsilon_k$, $d_2 = \varepsilon'_1\varepsilon'_2 \ldots \varepsilon'_j$, so that, in the notation of the algorithm, $P(d_1) = e_1e_2 \ldots e_k$ and $P(d_2) = e'_1e'_2 \ldots e'_j$.

If (i) is true then $e_i = e'_i$, $i = 1, 2, \ldots, m$, and, by Lemma 2, the string $e_1e_2 \ldots e_m e_{m+1}$ has an odd number e^-*'s* while the string $e'_1e'_2 \ldots e'_m e'_{m+1}$ has an even number. If $\varepsilon_m = \varepsilon'_m = 0$ then $e_{m+1} = e^+$, $e'_{m+1} = e^-$ and, by Lemma 3, $P(d_1) < P(d_2)$; similarly, if $\varepsilon_m = \varepsilon'_m = 1$, then $e_{m+1} = e^-$, $e'_{m+1} = e^+$ and, again by Lemma 3, $P(d_1) < P(d_2)$.

If (ii) is true, then $m = k < j$, $e_i = e'_i$, $i = 1, 2, \ldots, k$, and $\varepsilon'_k = 1$, so that the string $e'_1e'_2 \ldots e'_k$ has an even number of e^-'s. But since $0 < e'_{k+1}e'_{k+2} \ldots e'_j$, by Lemma 4, we have

$$P(d_1) = e_1e_2 \ldots e_k = e_1 \circ e_2 \circ \cdots \circ e_k(0)$$
$$< e_1 \circ e_2 \circ \cdots \circ e_k(e'_{k+1} \ldots e'_j) = e'_1e'_2 \ldots e'_j = P(d_2),$$

which completes the proof.

Lemma 5 makes possible a simple proof of continuity on the dyads in $(-1, 1)$ of any solution to (1) and (2). Since any such solution is strictly monotonic on the dyads we need merely show that for any given dyad, d in (0,1) there is an increasing and a decreasing sequence of dyads in (0, 1), both converging to d, whose images under P converge to $P(d)$. Let $d = .\varepsilon_1\varepsilon_2\varepsilon_3 \ldots \varepsilon_{k-1}1$ and, by the algorithm, $P(d) = .e_1e_2e_3 \ldots e_{k-1}e_k$. Let

$$\bar{e}_n = \begin{cases} e^-, & \text{if} \quad e_n = e^+ \\ e^+, & \text{if} \quad e_n = e^- \end{cases}$$

and let P_k denote the composite function $e_1 \circ e_2 \circ e_3 \circ \cdots \circ e_k$. Consider the sequence

$$\begin{aligned} d_1 &= .\varepsilon_1\varepsilon_2\varepsilon_3 \ldots \varepsilon_{k-1}01, & P(d_1) &= e_1e_2e_3 \ldots e_{k-1}\bar{e}_k e^-, \\ d_2 &= .\varepsilon_1\varepsilon_2\varepsilon_3 \ldots \varepsilon_{k-1}011, & P(d_2) &= e_1e_2e_3 \ldots e_{k-1}\bar{e}_k e^- e^+, \\ d_3 &= .\varepsilon_1\varepsilon_2\varepsilon_3 \ldots \varepsilon_{k-1}0111, & P(d_3) &= e_1e_2e_3 \ldots e_{k-1}\bar{e}_k e^- e^+ e^+, \\ d_4 &= .\varepsilon_1\varepsilon_2\varepsilon_3 \ldots \varepsilon_{k-1}01111, & P(d_4) &= e_1e_2e_3 \ldots e_{k-1}\bar{e}_k e^- e^+ e^+ e^+. \end{aligned}$$

Clearly $\{d_n\}$ is an increasing sequence whose limit is d. Moreover, since $\bar{e}_k(x) = e_k(-x)$, it follows that $P(d_1) = P_k(-e^-)$, $P(d_2) = P_k(-e^-e^+)$, $P(d_3) = P_k(-e^-e^+e^+)$, $P(d_4) = P_k(-e^-e^+e^+e^+)$, ... But the sequence $\{-e^-, -e^-e^+, -e^-e^+e^+, \ldots\}$ converges to 0 and P_k is a continuous function. Hence the sequence of images $\{P(d_n)\}$ converges to $P_k(0)=e_1e_2\dots e_k=P(d)$.

Similarly, consider the sequence

$$\begin{aligned}
D_1 &= .\varepsilon_1\varepsilon_2\varepsilon_3\dots\varepsilon_{k-1}11, & P(D_1) &= e_1e_2e_3\dots e_{k-1}e_ke^+,\\
D_2 &= .\varepsilon_1\varepsilon_2\varepsilon_3\dots\varepsilon_{k-1}101, & P(D_2) &= e_1e_2e_3\dots e_{k-1}e_ke^-e^-,\\
D_3 &= .\varepsilon_1\varepsilon_2\varepsilon_3\dots\varepsilon_{k-1}1001, & P(D_3) &= e_1e_2e_3\dots e_{k-1}e_ke^-e^+e^-,\\
D_4 &= .\varepsilon_1\varepsilon_2\varepsilon_3\dots\varepsilon_{k-1}10001, & P(D_4) &= e_1e_2e_3\dots e_{k-1}e_ke^-e^+e^+e^-,\\
D_5 &= .\varepsilon_1\varepsilon_2\varepsilon_3\dots\varepsilon_{k-1}100001, & P(D_5) &= e_1e_2e_3\dots e_{k-1}e_ke^-e^+e^+e^+e^-.
\end{aligned}$$

Clearly $\{D_n\}$ is a decreasing sequence of dyads whose limit is d. An argument similar to the one above shows that the sequence of images $\{P(D_n)\}$ is a sequence converging to $P(d)$. Thus any solution to (1) and (2) is continuous on the dyads in $(0, 1)$. Applying (1) and a similar argument to demonstrate continuity at zero, we get that any such solution is continuous on the dyads in $(-1, 1)$. Thus by continuity on a dense subset of $(-1, 1)$ we get that our unique solution of (1) and (2) on the dyads in $(-1, 1)$ can be extended to a unique solution on all of $(-1, 1)$. Thus we have proved

LEMMA 6. *The system of functional equations* (1) *and* (2) *possesses a unique, strictly increasing, continuous solution on the interval* $(-1, 1)$.

LEMMA 7. *The solution to* (1) *and* (2) *has a zero derivative at zero.*

Proof.[1]) Let $\{x_n\}$ be a strictly decreasing sequence converging to zero. For each x_n there exists a positive integer $m(n)$ such that

$$\left(\tfrac{1}{2}\right)^{m(n)+1} < x_n \leqslant \left(\tfrac{1}{2}\right)^{m(n)}$$

and thus

$$P\left(\left(\tfrac{1}{2}\right)^{m(n)+1}\right) < P(x_n) \leqslant P\left(\left(\tfrac{1}{2}\right)^{m(n)}\right)$$

with $\lim_{n\to\infty} m(n)=\infty$. Thus

$$\begin{aligned}
P'(0+) &= \lim_{n\to\infty}\left(\frac{P(x_n)-P(0)}{x_n-0}\right) = \lim_{n\to\infty}\left(\frac{P(x_n)}{x_n}\right) \leqslant \lim_{n\to\infty}\left(\frac{P\left(\left(\frac{1}{2}\right)^{m(n)}\right)}{\left(\frac{1}{2}\right)^{m(n)+1}}\right)\\
&= \lim_{n\to\infty}\frac{2^{m(n)+1}}{P\left(1-\left(\frac{1}{2}\right)^{m(n)}\right)} = 0,
\end{aligned}$$

since $P\left(1-\left(\frac{1}{2}\right)^{m(n)}\right)=\overbrace{e^+e^+\dots e^+}^{m(n)e^+\text{'s}}$ grows much faster then $2^{m(n)+1}$. Since P is an odd function, the same argument shows that $P'(0-)=0$, whence $P'(0)=0$.

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[1]) This argument is due to Prof. E. Killam.

LEMMA 8. *If* $x\in[0,1)$ *is such that* $P'(x)=0$ *then* $P'((x+1)/2)=P'(x/2)=0$.

Proof. This follows at once by differentiating (1) and (4).

LEMMA. 9 *If d is any dyad in* $(-1,1)$, *then* $P'(d)=0$.

Proof. The proof follows the lines of the proof that a solution to (1) and (2) was defined on the dyadic rationals but using Lemma 8 instead of (1) and (4).

This completes the proof of the Theorem.

We see then that P is a continuous monotonic bijection from the interval $(-1,1)$ to the reals. We may compute by (1) that $P(7/8)$ is approximately 15.1 and $P(15/16)$ is about 3.8×10^{6}. Thus, by applying first (3) and then (2), it follows that in the interval $(-1/16, 1/16)$ the function has absolute value less than 2.7×10^{-7}. Since the function is continuous and strictly monotone we might say that the function 'almost levels off' in an interval about zero. By (1) we also see that the function also 'almost levels off' in an interval about 1/2 and that the length of the interval in which it does so is a little less than half the length of the interval about zero. For example, we may easily compute that for $x\in(15/32, 17/32)$, $P(x)\in(\exp(-2.7\times10^{-7}),\ \exp(2.7\times10^{-7}))$. Thus, as the graph shows, the function is fairly 'level' at 1/2. Using (1) and (4) we further see that P, 'almost levels off' at each dyadic rational $m/2^n$ and that the width of the interval of leveling is, very approximately, proportional to $1/2^n$.

It has already been shown that $P'(0)=0$. The following argument will yield not only an alternate proof of this fact, but will also show just how 'flat' P actually is at zero.

From the estimate that $P(1/16)<2.7\times10^{-7}$ it follows that for any $u\in[1/64, 1/32]$:

$$P(2u)<10^{-6}<10^{-4}<u^2, \tag{6}$$

whence $1/P(2w)>10$, for $w\in(0, 1/32]$. Since $4v<e^v$ for $v>10$, it follows that for $w\in(0,1/32]$,

$$e^{-1/P(2w)}<P(2w)/4. \tag{7}$$

Now let $n\geqslant5$ and assume, as in (6), that $P(2x)<x^2$ for $x\in[1/2^{n+1}, 1/2^n]$. It follows by (7) that

$$P(x)=e^{-1/P(2x)}<P(2x)/4<x^2/4=(x/2)^2$$

and hence $P(x)<x^2$ for any $x\in[1/2^{n+2}, 1/2^{n+1}]$. This induction shows that $P(2x)<x^2$ for $x\in\bigcup_{n=5}^{\infty}[1/2^{n+1}, 1/2^n]=(0, 1/32]$. Whence, for any $x\in[-1/32, 1/32]$, $0\leqslant|P(x)|=P(|x|)=e^{-1/P(2|x|)}$.

But $e^{-1/x}$ is an increasing function so $e^{-1/P(2|x|)}\leqslant e^{-1/x^2}$. Thus we have shown

$$0\leqslant|P(x)|\leqslant e^{-1/x^2},\quad \text{for}\quad 0<x\leqslant1/32, \tag{8}$$

i.e. that $P(x)$ is even 'flatter' at zero than is e^{-1/x^2} a function notorious for being 'flat' at zero! Certainly

$$0 \leqslant P'(0) = \lim_{x \to 0} P(x)/x \leqslant \lim_{x \to 0} e^{-1/x^2}/x = 0, \quad \text{whence again} \quad P'(0) = 0.$$

III. Related Topics

The technique of this paper has been to begin with (1) and (2) and to first show that if P is restricted to $[0, 1)$, then (2) is equivalent to (4). In this case the system (1) and (4) is of the form

$$\left.\begin{aligned} f(x/2) &= g_0(f(x)) \\ f\left(\frac{x+1}{2}\right) &= g_1(f(x)) \end{aligned}\right\} \qquad (9)$$

where $g_0:(0, \infty) \to (0, 1)$ and $g_1:(0, \infty) \to (1, \infty)$ are strictly increasing homeomorphisms having unique fixed points at 0 and ∞, respectively. In the case we have studied $g_0(x) = e^{-1/x}$ and $g_1(x) = e^x$.

The system of functional equations (9), with $g_0(x) = x/(1+x)$ and $g_1(x) = 1+x$, has been previously considered by G. DeRham [1] who showed that for this pair of functions (9) has a unique, strictly monotone, continuous solution, having infinite derivative on each dyadic rational.

The fact that the same system of equations (9) arises in different contexts and has such interesting yet different solutions for different choices of g_0 and g_1, naturally leads one to study the system itself. As regards this project, we can report the following results:

Suppose f is a solution of the system (9) and that $f'(0)$ exists. Let

$$L = \lim_{n \to \infty} ((f(1/2^{n+1})/(1/2^{n+1}))/(f(1/2^n)/(1/2^n))) \qquad (10)$$

If $L < 1$, then $f'(0) = 0$; if $L > 1$, then $f'(0) = +\infty$; and if $L = 1$, this test is inconclusive. The quotient in (10) is equal to

$$2g_0(f(1/2^n))/f(1/2^n)$$

and, since g_0 is a homeomorphism whose only fixed point is zero, we get $\lim_{n \to \infty} f(1/2^n) = 0$. Thus, if it exists,

$$\lim_{v \to 0} 2g_0(v)/v = L.$$

For our case $g_0(x) = e^{-1/x}$, $\lim_{v \to 0} 2g_0(v)/v = 0$, and so P has zero derivative at zero. In DeRham's case $g_0(x) = x/(1+x)$, $\lim_{v \to 0} 2g_0(v)/v = 2$ and so his function has an infinite derivative at zero.

If a solution to (10) has infinite or zero derivative at zero and if g and g_1 have finite positive derivatives, then

$$f'(x/2) = 2g_0'(f(x))f'(x)$$
$$f'((x+1)/2) = 2g_1'(f(x))f'(x)$$

and the zero or infinite derivative of f at zero can be extended to each dyadic rational in the domain by arguments similar to those applied to P.

We have also considered the pair of functions g_0 and g_1 defined by

$$g_0(x) = \frac{-\alpha + \sqrt{\alpha^2 + 4x^2}}{2x}$$
$$g_1(x) = \frac{\alpha x + \sqrt{\alpha^2 x^2 + 4}}{2}$$

where $\alpha > 0$. Arguments similar to those used for the function P show that this system possesses a unique, strictly increasing, continuous solution. If $\alpha > 2$ the solution has derivative zero on the dyadic rationals. If $\alpha < 2$ the solution has infinite derivative on the dyadic rationals. In the indeterminate case, i.e., $\alpha = 2$. the solution is $\tan x\pi/2$, an analytic function!

In addition to the general study of (9) there are a number of specific open questions about P itself including the following:

(1) What values does $P'(x)$ take for x not a dyadic rational?

(2) Is the function P as 'flat' at each dyadic rational as e^{-1/x^2} is at zero?

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